JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **90**, 1–11 (1982)

Structural Stability of Nonchaotic Difference Equations

J. SMÍTAL AND K. SMÍTALOVÁ

Department of Mathematics, Komensky University, 84215 Bratislava, Czechoslovakia

Submitted by V. Lakshmikantham

1. Introduction

In this paper we shall deal with properties of solutions of the difference equation

$$x_{n+1} = f(x_n), \tag{1}$$

where $f: I \to I$ is a continuous function and I a compact-real interval. This equation is connected, e.g., with various biological phenomena (cf. [3, 5–7], among others). By a solution of (1) we mean a sequence

$$x, f(x), f^2(x), \ldots, \tag{2}$$

where $x \in I$ and f^i denotes the ith iterate of f. If for some $p \in I$ and some integer $n \geqslant 2$, $f^n(p) = p$, and $f^k(p) \neq p$ whenever $1 \leqslant k < n$, then the points $p, f(p), \ldots, f^{n-1}(p)$ form a cycle of order n. The behaviour of sequence (2) depends essentially on the existence of cycles of f. A possible approach in investigating such behaviour is based on results due to Šarkovskii [8, 9]. For each $x \in I$ there are the following three possible types of behaviour:

(i) Sequence (2) is convergent.

(ii) There is a point p of a cycle of some order such that $\lim_{n \to \infty} |f^n(x) - f^n(p)| = 0$. We say that x is asymptotically periodic. In this case sequence (2) has a finite number of cluster points and the cluster points form a cycle generated by p.

(iii) Sequence (2) has an infinite number of cluster points—it is chaotic.

It is known (cf. [9]) that for each $x \in I$ (2) is convergent iff f has no cycles. Further, for each $x \in I$ either (i) or (ii) holds if f has cycles of finite number of orders (which are necessarily of the form 2^n, cf. [8]). This

0022-247X/82/110001-11\$02.00/0

represents a highly desirable form of behaviour (f is nonchaotic). And finally, if f has a cycle of order which is not a power of 2, then the set of x for which (iii) holds has the power of the continuum (f is chaotic). The situation is complicated when f has cycles of infinitely many orders which are all of the form 2^n. Such functions can be chaotic or nonchaotic (cf. [9]). Concerning the notion of chaos, see [5] for a more precise definition.

Kloeden [4] has recently shown that the chaotic functions are dense in the space $\mathscr{C}(I)$ of continuous functions $I \to I$ with the max–norm. A simple modification of Kloeden's argument gives a somewhat stronger result: The nonchaotic functions form a nowhere-dense subset of $\mathscr{C}(I)$. This means that the nonchaotic equations (1) are structurally unstable, and the question arises as to whether such equations are suitable for describing real situations. The main aim of the present paper is to show that the situation is not so hopeless. Namely, if f is a function whose fixed points form a nowhere-dense set, if f has only cycles of finitely many different orders, and if g is a continuous function with $\|f-g\|$ sufficiently small then $\limsup_{n\to\infty} \|f^n - g^n\| < \varepsilon$, where $\varepsilon > 0$ is a given number (Theorem 3). This means that for g near to f the sequence x, $g(x)$, $g^2(x)$..., behaves nonchaotically up to small perturbations. It is also shown that the condition concerning fixed points of f cannot be omited (Theorem 1). For a stochastic approach to this problem see [1]. Note also that in [2, Theorem 2] the structural stability of chaotic functions is indicated.

The results are given in Section 2. Since the proof of the main theorem is rather complicated, we shall first present some lemmas in Section 3, then in Section 4 some preliminary constructions along with some additional lemmas, and finally in Section 5 a proof of the main theorem.

Concerning notation, I will always be a compact-real interval, and all functions will be continuous functions from I to I. For any function φ, and any positive integer n, φ^n denotes the nth iterate of φ. It is always understood that a cycle of φ means a cycle of order at least 2. The norm $\|\cdot\|$ of functions is always the max-norm.

2. Main Results

THEOREM 1. *Let f be a continuous function $I \to I$, and let $f(x) = x$ for $x \in [a, b] \subset I$, where $a < b$. Then for each $\delta > 0$ there is a continuous function g: $I \to I$ such that $\|f - g\| < \delta$ and*

$$\limsup_{k\to\infty} g^k(a) - \liminf_{k\to\infty} g^k(a) = b - a.$$

Proof. Let n be a positive integer. Denote $x_k = a + (k/n)(b - a)$, for

$k = 0, 1, \ldots, n$, and $y_k = b - ((2k-1)/2n)(b-a)$, for $k = 1, 2, \ldots, n$. Define g_n as follows:

$$g_n(x_k) = x_{k+1} \quad \text{for} \quad k = 0, 1, \ldots, n-1, \quad g_n(x_n) = y_1,$$
$$g_n(y_k) = y_{k+1} \quad \text{for} \quad k = 1, 2, \ldots, n-1, \quad g_n(y_n) = x_1.$$

Moreover, let g_n be linear on each of the intervals $[a, y_n]$, $[y_n, x_1]$, $[x_1, y_{n-1}], \ldots, [y_1, b]$. Clearly, $|g_n(x) - f(x)| = |g_n(x) - x| \leqslant (1/n)(b-a)$ for $x \in [a, b]$. Now extend g_n continuously to the whole I such that $\| g_n - f \| \leqslant (1/n)(b-a)$. It is easy to verify that for sufficiently large n, $g = g_n$ has the desired properties. ∎

The main result is the following:

THEOREM 2. *Let f be a continuous function $I \to I$. Assume that f has no cycles and that the set of fixed points of f is nowhere dense in I. Then for each $\varepsilon > 0$ there is some $\delta > 0$ such that for each continuous $g: I \to I$ with $\| f - g \| < \delta$, and for each $x \in I$,*

$$\limsup_{n \to \infty} g^n(x) - \liminf_{n \to \infty} g^n(x) < \varepsilon. \tag{3}$$

Remark. The set A of fixed points of f is closed, hence A is nowhere dense iff A contains no interval. Theorem 1 thus shows that the nowhere density of A is essential in Theorem 2.

As a consequence of Theorem 2 we obtain a more general

THEOREM 3. *Let f: $I \to I$ be a continuous function. Assume that f has only cycles of order $\leqslant 2^n$. Moreover, assume that both the set of fixed points of f and the set of cyclic points of f are nowhere-dense sets. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each continuous $g: I \to I$ with $\| f - g \| < \delta$,*

$$\limsup_{n \to \infty} \| f^n - g^n \| < \varepsilon$$

Proof. First we show that for each positive integer m the mapping $\Phi_m: \varphi \to \varphi^m$ is a continuous operator on the space $\mathscr{C}(I)$ of continuous functions $I \to I$. Assume by induction that Φ_m is continuous. Then

$$\begin{aligned} \| f^{m+1} - g^{m+1} \| &\leqslant \| f \circ f^m - f \circ g^m \| + \| f \circ g^m - g \circ g^m \| \\ &\leqslant \omega_f \| f^m - g^m \| + \| f - g \|, \end{aligned} \tag{4}$$

where ω_f is the modulus of continuity of f. Now when $f \to g$, then $f^m \to g^m$ by hypothesis, and the right-hand side of (4) tends to 0.

Now f has only cycles whose orders are divisors of 2^n (cf. [8]), hence, $\varphi = f^{2^n}$ satisfies the hypothesis of Theorem 2. Each f^m can be represented as $f^k \circ \varphi^s$, where $0 \leqslant k < 2^n$, and similarly $g^m = g^k \circ \psi^s$, where $\psi = g^{2^n}$. Hence, for a fixed $k = 0, 1, \ldots, 2^n - 1$ we have

$$\limsup_{s \to \infty} \|f^k \circ \varphi^s - g^k \circ \psi^s\| \leqslant \omega_{f^k} (\limsup_{s \to \infty} \|\varphi^s - \psi^s\|) + \|f^k - g^k\| = \alpha_k(g),$$

and hence, Theorem 2 and the continuity of Φ_k imply that $\lim \alpha_k(g) = 0$ whenever $g \to f$. Consequently, $\limsup \|f^n - g^n\| < \varepsilon$. ∎

3. Preliminary Lemmas

We recall that φ is always a continuous function $I \to I$. In what follows we do not distinguish between a function or a relation and its graph.

Lemma 1. *Let φ^{-1} be the inverse relation of a function φ(φ^{-1} need not be a function). For each $X \subset I$ denote*

$$X_\varphi = \varphi \cap (X \times I), \qquad X_\varphi^{-1} = \varphi^{-1} \cap (X \times I).$$

If φ contains no cycle and if X is a closed set which contains no fixed point of φ, then

$$\operatorname{dist}(X_\varphi, X_\varphi^{-1}) > 0.$$

Proof. It is easy to see that X_φ and X_φ^{-1} are compact sets. If $\operatorname{dist}(X_\varphi, X_\varphi^{-1}) = 0$ for some X, then X_φ and X_φ^{-1} have a common point $z = \langle x, y \rangle$. Since $z \in X_\varphi$, we have $\varphi(x) = y$, and since $z \in X_\varphi^{-1}$, $\varphi(y) = x$. Since $x \in X$, $x \neq \varphi(x)$, hence, $x \mapsto y \mapsto x$ is a 2-cycle of φ, which is impossible.

For each nonnegative λ define

$$A_\varphi(\lambda) = \{x \in I; |\varphi(x) - x| \leqslant \lambda\}.$$

Clearly, each $A_\varphi(\lambda)$ is a closed set. ∎

Lemma 2. *Let $0 \leqslant \lambda_1 < \lambda_2$. Then there is a finite number of connected components (intervals) $M_1, \ldots, M_m$ of the set $I \backslash A_\varphi(\lambda_1)$ such that for each i, $M_i \backslash A_\varphi(\lambda_2) \neq \varnothing$.*

Proof. Assume that $M_1, M_2, \ldots,$ is an infinite sequence of such components which are pairwise disjoint. Let a be a cluster point of this sequence. Then each neighborhood $O(a)$ of a contains an interval M_n, and hence, points $x, y \in M_n$ such that $\varphi(x) = \lambda_1$, $\varphi(y) > \lambda_2$. Consequently φ is discontinuous at a, which is a contradiction. ∎

The following lemma is implicitly contained, e.g., in [9], but for completeness we present its proof.

LEMMA 3. *Assume that there are $x, y \in I$, $x < y$ such that $\varphi(x) \geqslant y$ and $\varphi(y) \leqslant x$. Then φ contains a 2–cycle.*

Proof. If $\varphi(x) = y$ and $\varphi(y) = x$, then φ has a 2-cycle. So assume that $\varphi(y) < x$. Since $\varphi(x) > x$ and $\varphi(y) < y$, there is a fixed point of φ between x and y; denote it by z. Since $[z, y] \subset \varphi[x, z]$, there is some $y_1 \in [x, z)$ with $\varphi(y_1) = y$. Denote by α the left-hand endpoint of I. Then $\varphi^2(\alpha) \geqslant \alpha$, $\varphi^2(y_1) < x \leqslant y_1$, hence, there is a greatest fixed point v of φ^2 in (α, y_1). It suffices to assume that $\varphi(v) = v$ since otherwise φ has a cycle. So $[v, z] \subset \varphi[v, y_1]$, hence for some $z_1 \in (v, y_1)$, $\varphi(z_1) = z$. Then $\varphi^2(z_1) = z > y_1 > z_1$ and $\varphi^2(y_1) < y_1$. Thus φ^2 has a fixed point between z_1 and y_1, which is impossible. ∎

4. Preliminary Constructions

Fix some $\varepsilon > 0$ and a function f satisfying the assumptions of Theorem 2.

Choose $\delta_1 > 0$ such that $A_f(\delta_1)$ contains no interval of the length $\geqslant \varepsilon/6$ (see Lemma 2). Such a δ_1 exists since $A_f(0)$ is nowhere dense. Then there are connected components $I_1, \ldots, I_k$ of the set $I \setminus A_f(\delta_1)$ such that the set

$$I \setminus (I_1 \cup \cdots \cup I_k) \text{ contains no interval of the length } \geqslant \varepsilon/3. \tag{5}$$

Denote by δ_2 the length of the smallest interval I_i. Choose δ_3 such that

$$0 < \delta_3 < \min\{\delta_1, \delta_2, \varepsilon/4\} \tag{6}$$

and let $J_1, \ldots, J_m$ be all of the connected components of the set $I \setminus A_f(\delta_3/3)$ which are not contained in $A_f(2\delta_3/3)$ (see Lemma 2). Clearly, $m \geqslant k$ and each I_i is contained in some J_j. Denote $J = J_1 \cup \cdots \cup J_m$. Then the set J has the following property:

LEMMA 4. *Let $\|f - g\| \leqslant \delta_3/3$, and assume that for some $x \in I$, $|g(x) - x| \geqslant \varepsilon/4$, or that between x and $g(x)$ there lies an interval I_i. Then $x \in J$.*

Proof of the lemma. In both cases $|g(x) - x| \geqslant \delta_3$, hence $|f(x) - x| \geqslant |g(x) - x| - |f(x) - g(x)| > \delta_3 - \delta_3/3$, i.e., $x \in I \setminus A_f(2\delta_3/3) \subset J$. ∎

Now choose a δ_4 such that

$$0 < \delta_4 < \delta_3/3 \tag{7}$$

and such that the closed δ_4-neighbourhood S of J is disjoint with $A_f(0)$. Then $f^{-1}(S) \cap A_f(0) = \varnothing$. Denote

$$\delta_5 = \min\{|f(x) - x|;\ x \in f^{-1}(S)\}.$$

Since $f^{-1}(S)$ and $A_f(0)$ are disjoint compact sets we have $\delta_5 > 0$. Let δ_6 be the length of the smallest interval J_i. Choose δ_7 such that

$$0 < \delta_7 < \min\{\delta_4, \delta_5, \delta_6\} \tag{8}$$

and let $K_1, \ldots, K_s$ be the connected components of $I \setminus A_f(\delta_7/3)$ which are not contained in $A_f(2\delta_7/3)$ (see Lemma 2). Clearly, each J_i is contained in some K_j. Denote $K = K_1 \cup \cdots \cup K_s$. The set K has the following property:

LEMMA 5. *Let $\|f - g\| < \delta_7/3$. Assume that for some x, $g(x) \in J$, or that between x and $g(x)$ there lies some interval J_i. Then $x \in K$.*

Proof of the lemma. If $g(x) \in J$, then $f(x) \in S$, so $x \in f^{-1}(S) \subset I \setminus A_f(\delta_5) \subset I \setminus A_f(2\delta_7/3) \subset K$. If some J_i lies between x and $g(x)$, then $|g(x) - x| \geqslant \delta_6 > \delta_7$, hence $|f(x) - x| \geqslant |g(x) - x| - |f(x) - g(x)| > 2\delta_7/3$, and again $x \in K$. ∎

Denote by L_i ($i = 1, \ldots, s$) the connected component of $I \setminus A_f(\delta_7/4)$ which contains K_i, and let L be the closure of $L_1 \cup \cdots \cup L_s$. Clearly,

$$L \supset K \supset J \supset I_1 \cup \cdots \cup I_k. \tag{9}$$

Let

$$\delta_8 = \operatorname{dist}(L_f, L_f^{-1}) \tag{10}$$

(see Lemma 1). Finally choose δ such that

$$0 < \delta < \min\{\delta_8/2, \delta_7/3\} \tag{11}$$

and fix some g such that $\|f - g\| < \delta$. Since $\delta < 2\delta_7/3$ and since $L \subset I \setminus A_f(\delta_7/4)$, we have

$$g(x) \neq x \qquad \text{for} \quad x \in L. \tag{12}$$

Now we show that (3) holds for each $x \in I$. First we introduce the following auxiliary function h: Define

$$h(x) = f(x) \qquad \text{for} \quad x \in I \setminus L,$$

$$h(x) = g(x) \qquad \text{for} \quad x \in K,$$

and extend h continuously onto the whole I such that

$$\|h-f\|<\delta \qquad \text{and} \qquad \|h-g\|<\delta. \tag{13}$$

Such an extension is possible since the set $L\backslash K$ consists of at most $2s$ intervals. We show that h has no cycle. Assume the contrary. Then h has a 2-cycle $x_1 \mapsto x_2 \mapsto x_1$, $x_1 \neq x_2$ (cf. [8]). At least one x_i, say x_1 must belong to L since $f(x)=h(x)$ for $x \notin L$ and f has no cycles. But in this case $\operatorname{dist}(L_h, L_h^{-1})=0$ since $z=\langle x_1, x_2\rangle \in L_h \cap L_h^{-1}$ (see Lemma 1). Therefore $\operatorname{dist}(z, L_f) \leqslant \delta$, $\operatorname{dist}(z, L_f^{-1}) \leqslant \delta$, and hence dist $(L_f, L_f^{-1}) \leqslant \delta+\delta<\delta_8$, which is a contradiction (see (10) and (11)). Hence we have proved the following:

LEMMA 6. *There is a continuous function h which has no cycles and such that* $\|f-h\|<\delta$, $\|g-h\|<\delta$, and $h(x)=g(x)$ for $x \in K$.

5. Proof of Theorem 2

Assume that (3) does not hold for some $x \in I$. Denote

$$g^n(x)=x_n, \qquad \liminf_{n\to\infty} x_n=a, \qquad \limsup_{n\to\infty} x_n=b.$$

So we have $b-a \geqslant \varepsilon$. From (5) it follows that for some i, $I_i \cap (a,b) \neq \varnothing$. Consider the following two possible cases:

Case (a).

There is a $j \leqslant k$ such that the closure $\bar{I}_j$ of I_j is contained in the interval (a,b).

Case (b).

Case (a) does not hold.

First we consider Case (a). By (12) and (9), $g(t)-t$ does not change the sign for $t \in I_j$. By symmetry we may assume that

$$g(t)<t \qquad \text{for} \quad t \in \bar{I}_j. \tag{14}$$

Consider the following three subcases (a1)–(a3) of Case (a).

Subcase (a1). There is some integer $n=n(1)$ with $x_{n(1)} \in \bar{I}_j$. Let $n(2)>n(1)$ be the least number with the property

$$x_{n(2)}<x_{n(2)+1} \qquad \text{and} \qquad \bar{I}_j \subset (x_{n(2)}, x_{n(2)+1}).$$

The existence of such $n(2)$ follows from the fact that a, b are cluster points and that $\bar{I}_j \subset (a, b)$. Let $n(3)$ be the greatest index $< n(2)$ with $x_{n(3)} > x_{n(2)}$. Clearly, $n(3) \geqslant n(1)$ and, therefore (see also (14) and the definition of $n(2)$),

$$x_{n(2)+1} > x_{n(3)} > x_{n(2)} \geqslant x_{n(3)+1}. \tag{15}$$

By Lemma 4, $x_{n(2)}$ belongs to some J_i and from (12) we have $g(t) > t$ for $t \in J_i$, hence $x_{n(3)} \notin J_i$, and hence by Lemma 5, $x_{n(3)} \in K$. Now from (15) and from the fact that $h(x_{n(2)}) = g(x_{n(2)})$ and $h(x_{n(3)}) = g(x_{n(3)})$ it follows by Lemma 3 that h has a cycle, which contradicts Lemma 6.

In the Cases (a2) and (a3) $\bar{I}_j$ contains no point x_n. Define sets X, Y by

$$x_n \in X \quad \text{iff} \quad x_n < x_{n+1} \quad \text{and} \quad \bar{I}_j \subset (x_n, x_{n+1}),$$

$$x_n \in Y \quad \text{iff} \quad x_{n+1} < x_n \quad \text{and} \quad \bar{I}_j \subset (x_{n+1}, x_n).$$

Both the sets X, Y are infinite since a, b are cluster points. Let $x_{n(1)}, x_{n(3)}, x_{n(5)}, \ldots,$ be the enumeration of all members of X such that $n(1) < n(3) < n(5) < \cdots$ and let $x_{n(2)}, x_{n(4)}, x_{n(6)}, \ldots$ be the enumeration of all members of Y with $n(2) < n(4) < n(6) < \cdots$. Since no x_n belongs to I_j, $\{n(\mathrm{i})\}_{j=1}^{\infty}$ must be an increasing sequence.

Subcase (a2). There is some $p \geqslant 1$ such that

$$x_{n(2p+1)} < x_{n(2p)+1} \tag{16}$$

or

$$x_{n(2p-1)+1} < x_{n(2p)}. \tag{17}$$

Because of symmetry it suffices to consider (16). Let n be the greatest number $< n(2p+1)$ with the property $x_n > x_{n(2p+1)}$. Such an n clearly exists since we have $n \geqslant n(2p) + 1$ and x_n lies to the left of $\bar{I}_j$. Then $x_{n+1} \leqslant x_{n(2p+1)}$. But $x_{n(2p+1)}$ must belong to some interval J_i (Lemma 4) with the property $g(t) > t$ for $t \in J_i$ (see (12)), and $g(x_n) < x_n$, hence $x_n \notin J_i$. If $x_{n+1} \in J_i$, then $x_n \in K$ (Lemma 5). If $x_{n+1} \notin J_i$ then $J_i \subset (x_{n+1}, x_n)$, and hence again by Lemma 5, $x_n \in K$. Consequently, x_n, $x_{n(2p+1)} \in K$, $x_n > x_{n(2p+1)}$, by Lemma 6, $h(x_n) \leqslant x_{n(2p+1)}$, $h(x_{n(2p+1)}) > x_n$, and by Lemma 3, h has a 2-cycle which is impossible.

Subcase (a3). For each i,

$$x_{n(2i+1)} \geqslant x_{n(2i)+1} \tag{18}$$

and

$$x_{n(2i)} \leqslant x_{n(2i-1)+1}. \tag{19}$$

Assume that for some $p \geqslant 1$,

$$x_{n(2p+1)+1} \geqslant x_{n(2p)} \tag{20}$$

or

$$x_{n(2p)+1} \leqslant x_{n(2p-1)}. \tag{21}$$

Then by Lemma 3, h has a cycle: Indeed, when (20) holds, then

$$x_{n(2p)}, x_{n(2p+1)} \in K, \qquad x_{n(2p)} > x_{n(2p+1)}, \qquad h(x_{n(2p+1)}) \geqslant x_{n(2p)},$$

and by (18) $h(x_{n(2p)}) \leqslant x_{n(2p+1)}$. When (21) holds the argument is similar.

It remains to check the case when for each i inequalities (18), (19) are satisfied along with

$$x_{n(2i+1)+1} < x_{n(2i)} \tag{22}$$

and

$$x_{n(2i)+1} > x_{n(2i-1)}. \tag{23}$$

Then we have, by (18), (23), (19), (22)

$$\begin{aligned} x_{n(2)+1} \leqslant x_{n(3)} < x_{n(4)+1} \leqslant x_{n(5)} < \cdots < \alpha < \beta \\ < \cdots < x_{n(4)} \leqslant x_{n(3)+1} < x_{n(2)} \leqslant x_{n(1)+1} \end{aligned}$$

where α, β are the endpoints of I_j. Denote

$$\lim_{i\to\infty} x_{n(2i+1)} = c, \qquad \lim_{i\to\infty} x_{n(2i)} = d.$$

Clearly, $c < d$ and by the continuity of g, $g(c) = d$ and $g(d) = c$. Since $I_j \subset (c, d)$, by Lemma 4, c, $d \in J \subset K$. Hence by Lemma 6, h has a 2-cycle, $h(c) = d$, $h(d) = c$, which is impossible.

Case (b)

Since (a) does not hold, by (5) there must be a $j \leqslant k$ such that $I_j \cap (a, b)$ is an interval of the length at least $\varepsilon/3$, and such that $a \in \bar{I}_j$ or $b \in \bar{I}_j$. Without loss of generality we may assume that $b \in \bar{I}_j$. We show that in this case $a \notin \bar{I}_j$. It is easy to see that $g(a)$, $g(b)$ are cluster points of $\{x_n\}_{n=1}^{\infty}$, hence $g(a) \geqslant a$ and $g(b) \leqslant b$. Since $b \in \bar{I}_j \subset L$, (12) implies that $g(b) < b$, hence $g(t) < t$ for $t \in \bar{I}_j$, and hence $a \notin \bar{I}_j$. Denote the left-hand endpoint of I_j by c. Let V be the set of those x_n, for which $x_n < c$ and $x_{n+1} > c + \varepsilon/4$.

Clearly, V is an infinite set. This follows from the fact that $c + \varepsilon/4 < b$, b is a cluster point, and

$$g(t) < t \qquad \text{for} \quad t \in [c, b]. \tag{24}$$

Now let $x_{n(1)}, x_{n(2)},...,$ be the enumeration of V such that $n(1), n(2),...,$ is an increasing sequence. Consider Subcases (b1) and (b2).

Subcase (b1). For some p, $x_{n(p+1)} < x_{n(p)}$.

Let n be the greatest number $< n(p+1)$ with $x_n > x_{n(p)}$ (such n clearly exists and $n \geqslant n(p) + 1$). Then $x_{n+1} \leqslant x_{n(p)}$ and by (24) $x_n \leqslant x_{n(p)+1}$. Since $x_{n(p)+1} - x_{n(p)} > \varepsilon/4$, by Lemma 4 $x_{n(p)} \in J$, and hence, by Lemma 5, $x_n \in K$. Now similarly as in the preceding cases $x_{n(p)} < x_n$, $h(x_{n(p)}) \geqslant x_n$, $h(x_n) \leqslant x_{n(p)}$, and hence, by Lemma 3, h has a 2–cycle.

Subcase (b2). We have

$$x_{n(1)} \leqslant x_{n(2)} \leqslant x_{n(3)} \leqslant \cdots \leqslant c.$$

Let

$$d = \lim_{i \to \infty} x_{n(i)}. \tag{25}$$

Then by the continuity of g, $g(d) - d \geqslant \varepsilon/4$, hence by Lemma 4, $d \in J$.

If $g^2(d) \leqslant d$, then again by Lemma 4, $g(d) \in J$ and similarly as in the preceeding case, h has, by Lemma 3, a 2-cycle. Otherwise, $d < g^2(d) < g(d)$. Then (25) implies that for some sufficiently large v we have

$$x_{n(v)} \leqslant x_{n(v+1)} \leqslant d < x_{n(v)+2} < x_{n(v)+1}.$$

Now let n be the greatest number $< n(v+1)$ such that $x_n > d$ (clearly, $n \geqslant n(v) + 2$). Then $x_{n+1} \leqslant d$, d, $x_{n+1} \in K$, and Lemma 3 implies that h has a 2-cycle, which is a contradiction. This completes the proof of Theorem 2.

REFERENCES

1. A. BOYARSKY, Randomness implies order, *J. Math. Anal. Appl.* **76** (1980), 483–497.
2. G. J. BUTLER AND G. PIANIGIANI, Periodic points and chaotic functions in the unit interval, *Bull. Austral. Math. Soc.* **18** (1978), 255–265.
3. K. L. COOKE, D. F. CALEF, AND E. V. LEVEL, Stability or chaos in discrete epidemic systems, *in* "Proceedings of the Conference on Nonlinear Systems and Applications," Arlington, Acacdemic Press New York, 1977.
4. P. E. KLOEDEN, Chaotic difference equations are dense, *Bull. Austral. Math. Soc.* **15** (1976), 371–379.

5. T. Y. LI AND J. A. YORKE, Period three implies chaos, *Amer. Math. Monthly* **82** (1975), 985–992.
6. R. M. MAY, Biological populations with nonoverlapping generations: stable points, stable cycles, and chaos, *Science* **186** (1974), 645–647.
7. R. M. MAY, Biological populations obeying difference equations: stable points, stable cycles, and chaos, *J. Theoret. Biol.* **51** (1975), 511–524.
8. A. N. ŠARKOVSKII, Coexistence of cycles of a continuous transformation of a line into itself, *Ukrain. Mat. Ž.* **16** (1) (1964), 61–71 (in Russian).
9. A. N. ŠARKOVSKII, On cycles and the structure of a continuous transformation, *Ukrain. Mat. Ž.* **17** (3) (1965), 104–111 (in Russian).